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Letter to the Editor

A preliminary study of the Irving–Mullineux nonlinear oscillator equation

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The following non-linear differential equation was introduced by Irving and Mullineux [1] as an example for which the perturbation method could be applied:

$$\frac{d^2x}{dt^2} + x = -\varepsilon\left[\frac{dx}{dt} + \left(\frac{dx}{dt}\right)^{2/3}\right]. \tag{1}$$

Under the assumption that the parameter ε was both positive and small, they wrote

$$x(t) = x_0(t) + \varepsilon x_1(t) + O(\varepsilon^2), \tag{2a}$$

$$x(0) = A, \quad \dot{x}(0) = 0 \tag{2b}$$

and obtained the results

$$x_0(t) = A\cos t, \tag{3a}$$

$$x_1(t) = A\left\{(\sin t - t\cos t)/2 - \left(\frac{1}{A^{1/3}}\right)\left[\left(\frac{3}{5}\right)(\sin t)^{8/3} - (\cos t)\int_0^t (\sin\theta)^{5/3}\,d\theta\right]\right\} \tag{3b}$$

Inspection of Eq. (3) shows that based on their calculations the obtained perturbation solution is not periodic to terms of order ε. This follows from the fact that $x_1(t)$ contains secular terms [2] and, consequently, is not a uniformly valid approximation [2]. Further, this particular calculation implicitly assumes that the solutions to Eq. (1) are not only oscillatory, but also periodic. This is not likely to be the actual situation since Eq. (1) contains a linear damping term on its right-side.

The major purpose of this communication is to examine Eq. (1) somewhat more detailed than Irving and Mullineux [1] and try to come to some conclusions regarding the general behavior of its solutions. To do this, this equation is first examined from the point of view of a general "physical equation" where all the variables and parameters have physical units [3]. It will be shown that two possible dimensionless equations can be constructed. Secondly, a study of the system equations in phase-space will be done; however, the preliminary analysis does not provide much information

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except for the existence of a single fixed-point or equilibrium solution at $(\bar{x}, \bar{y}) = (0, 0)$. Finally, a numerical integration scheme [4] will be used to examine the solutions to Eq. (1) for a wide range of initial values and magnitudes of the parameter ε.

It should be noted that the right side of Eq. (1) contains a derivative term raised to a fractional power. This is the major source of the difficulties that occurred in Ref. [1]. Such a term leads to expressions having negative powers when the standard (regular) perturbation methods are applied [1].

To begin, rewrite Eq. (1) in the physical equation form

$$\frac{d^2y}{d\bar{t}^2} + k_1 \frac{dy}{d\bar{t}} + k_2 y = -k_3 \left(\frac{dy}{d\bar{t}}\right)^{2/3}, \tag{4}$$

where all the parameters are taken to be positive and where y and $\bar{t}$ have the physical units, respectively, of distance (L) and time (T). Denoting the physical units of a variable or parameter by the symbol $[\cdots]$, it follows that

$$[y] = L, \quad [\bar{t}\,] = T, \quad [k_1] = \frac{1}{T}, \quad [k_2] = \frac{1}{T^2}, \quad [k_3] = \frac{L^{1/3}}{T^{4/3}}. \tag{5}$$

Based on these results, the following two time and length scales can be constructed [2,3]:

$$T_1 = \frac{1}{\sqrt{k_2}}, \quad T_2 = \frac{1}{k_1}, \tag{6a}$$

$$L_1 = \frac{k_3^3}{k_2^2}, \quad L_2 = \frac{k_3^3}{k_1^4}. \tag{6b}$$

Observe that T_1 and T_2 are, respectively, related to the periods of free, undamped harmonic oscillator and the damped, linear harmonic oscillator [2].

Using the scaled variables

$$\bar{t} = T_1 t \quad \text{and} \quad y = L_2 x, \tag{7}$$

where (t, x) are dimensionless variables, Eqn. (4) takes the form

$$\frac{d^2x}{dt^2} + 2\varepsilon \frac{dx}{dt} + x = -(2\varepsilon)^{4/3} \left(\frac{dx}{dt}\right)^{2/3}, \tag{8}$$

where

$$2\varepsilon = \frac{T_1}{T_2}. \tag{9}$$

Observe that the parameter ε occurs in both the first- and $\frac{4}{3}$-power forms. This minor problem can be overcome by the transformation

$$x = \varepsilon^{\beta} u, \quad \beta > 0. \tag{10}$$

Substituting this into Eq. (8) gives

$$\frac{d^2u}{dt^2} + 2\varepsilon \frac{du}{dt} + u = -(2\varepsilon)^{\alpha} \left(\frac{du}{dt}\right)^{2/3}, \tag{11}$$

where

$$\alpha \equiv \frac{4}{3} + \left(\frac{2}{3}\right)\beta - \beta. \tag{12}$$

The requirement that $\alpha = 1$ gives $\beta = 1$, and Eq. (11) becomes

$$\frac{d^2u}{dt^2} + u = -2\varepsilon\left[\frac{du}{dt} + 2^{1/3}\left(\frac{du}{dt}\right)^{2/3}\right], \tag{13}$$

which is of exactly the same form as Eq. (1) except for the labelling of the dependent variable, and a factor of two appears with ε.

The appearance of ε in Eq. (11) makes it difficult or maybe impossible to formulate a perturbation method to "solve" it. Clearly, the form given in Eq. (13) is much better suited for the construction of a perturbation procedure. For the remainder of this paper, the form of the I–M equation given in Eq. (13) will be used with the replacements $u = x$ and $du/dt = dx/dt = y$.

The system equations for Eq. (13) are

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x - 2^{4/3}\varepsilon y - 2\varepsilon y^{2/3} \tag{14}$$

and the trajectories in the (x, y) phase-space are solutions to the first order differential equation

$$\frac{dy}{dx} = -\frac{x + 2\varepsilon y + 2^{4/3}\varepsilon y^{2/3}}{y}. \tag{15}$$

From Eq. (14) it follows that there is only one fixed-point or equilibrium solution and it is located at the origin of the phase-space, i.e., $(\bar{x}, \bar{y}) = (0, 0)$. However, the calculation and plotting of the nullclines for Eq. (15) does not lead to any definitive conclusion regarding the stability of the fixed-point [5]. In other words, the geometrical properties of the nullclines in a neighborhood of the origin is consistent with the fixed-point being either stable or unstable.

One possibility to resolve this difficulty is to use the method of slowly varying amplitude and phase (MSVAP) [2]. To first order in the small parameter ε the amplitude $a(t, \varepsilon)$ and phase $\phi(t, \varepsilon)$ are given by

$$\frac{da}{dt} = \left(\frac{\varepsilon}{2\pi}\right)\int_0^{2\pi} f(a\sin\psi, a\cos\psi)\cos\psi \, d\psi, \tag{16a}$$

$$\frac{d\phi}{dt} = \left(\frac{\varepsilon}{2\pi a}\right)\int_0^{2\pi} f(a\sin\psi, a\cos\psi)\sin\psi \, d\psi, \tag{16b}$$

where for Eq. (13)

$$f(x, \dot{x}) = -2\dot{x} - 2^{4/3}(\dot{x})^{2/3} = -2a\cos\psi - 2^{4/3}(a\cos\psi)^{2/3} \tag{17}$$

and the solution, $x(t, \varepsilon)$, is given by

$$x(t, \varepsilon) = a(t, \varepsilon)\sin[t + \phi(t, \varepsilon)]. \tag{18}$$

Note that as a function of ψ, the function f is periodic with period 2π and even. Consequently, the integral on the right side of Eq. (16b) is zero. Therefore,

$$\frac{d\phi}{dt} = 0 \quad \text{or} \quad \phi(t, \varepsilon) = \phi_0 = \textit{constant}. \tag{19}$$

Also, observe that $(\cos\psi)^{2/3}$ is periodic with period 2π, is non-negative, and its Fourier expansion takes the form [6]

$$(\cos\psi)^{2/3} = \frac{f_0}{2} + \sum_{k=1}^{\infty} f_k \cos(2k\psi). \tag{20}$$

Using this result, the integral on the right side of Eq (16a) is trivial and the following is obtained:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\varepsilon a \tag{21}$$

for which the solution is

$$a(t,\varepsilon) = a_0 \mathrm{e}^{-\varepsilon t}, \quad a_0 = constant. \tag{22}$$

Hence, the MSVAP gives as an approximation to the solution of Eq. (13) the expression

$$x(t,\varepsilon) = a_0 \mathrm{e}^{-\varepsilon t} \sin(t+\phi_0). \tag{23}$$

This shows that all the solutions to Eq. (13) oscillate with the amplitude decreasing as a function of time. From this analysis, it follows that the fixed-point, $(\bar{x},\bar{y}) = (0,0)$, is stable.

Before leaving this particular calculation, it should be indicated that while the MSVAP can be easily applied to Eq. (13) to determine a solution, to terms of order ε [2], this procedure cannot be used to generate higher order in ε corrections to this lowest-order result. The reason for this limitation is the occurrence of the $(\dot{x})^{2/3}$ term on the right side of Eq. (13).

An important technique used extensively in modern scientific and engineering research is the application of numerical techniques to solve differential equations arising from the mathematical modelling of various phenomena [7,8]. A nonstandard finite difference scheme [4] for Eqs. (14) is

$$\frac{x_{k+1} - \psi x_k}{\phi} = y_k, \tag{24a}$$

$$\frac{y_{k+1} - \psi y_k}{\phi} = -x_k - 2\varepsilon y_k - 2^{4/3}\varepsilon (y_k)^{2/3}, \tag{24b}$$

where $t_k = hk$, $h = \Delta t$; x_k and y_k are, respectively, approximations to $x(t_k)$ and $y(t_k)$; and the functions $\psi(\varepsilon,h)$ and $\phi(\varepsilon,h)$ are

$$\psi(\varepsilon,h) = \frac{\varepsilon \mathrm{e}^{-\varepsilon h}}{\sqrt{1-\varepsilon^2}} \sin\left[\left(\sqrt{1-\varepsilon^2}\right)h\right] + \mathrm{e}^{-\varepsilon h} \cos\left[\left(\sqrt{1-\varepsilon^2}\right)h\right], \tag{25a}$$

$$\phi(\varepsilon,h) = \frac{\mathrm{e}^{-\varepsilon h}}{\sqrt{1-\varepsilon^2}} \sin\left[\left(\sqrt{1-\varepsilon^2}\right)h\right]. \tag{25b}$$

Inspection of Eqs. (24) shows that this scheme is explicit, i.e., x_{k+1} and y_{k+1} can be expressed as functions of $(x_k, y_k, \varepsilon, h)$. Using the relation for y_k, given in Eq. (24a), a single second order difference equation can be obtained for x_k; it is

$$\frac{x_{k+1} - 2\psi x_k + \psi^2 x_{k-1}}{\phi^2} + 2\varepsilon\left(\frac{x_k - x_{k-1}}{\phi}\right) + x_{k-1} = -2^{4/3}\varepsilon\left(\frac{x_k - \psi x_{k-1}}{\phi}\right)^{2/3}. \tag{26}$$

Using the scheme in Eqs. (24), a large number of numerical solutions were evaluated for various values of the initial conditions, x_0 and y_0, step-size h, and parameter ε for $0 \leqslant \varepsilon < 1$. Plots of x_k and y_k versus t_k, and phase-space, y_k versus x_k, were made and studied. The general features obtained were (1) both x_k and y_k had damped oscillatory behavior with

$$\lim_{t_k \to \infty} (x_k, y_k) = (0, 0); \tag{27}$$

(2) the trajectories in the (x_k, y_k) phase-space all spiraled into the origin. A typical set of plots is given in Fig. 1 for the parameter values $x_0 = 10$, $y_0 = 0.01$, $h = 0.001$ and $= 0.01$. Of

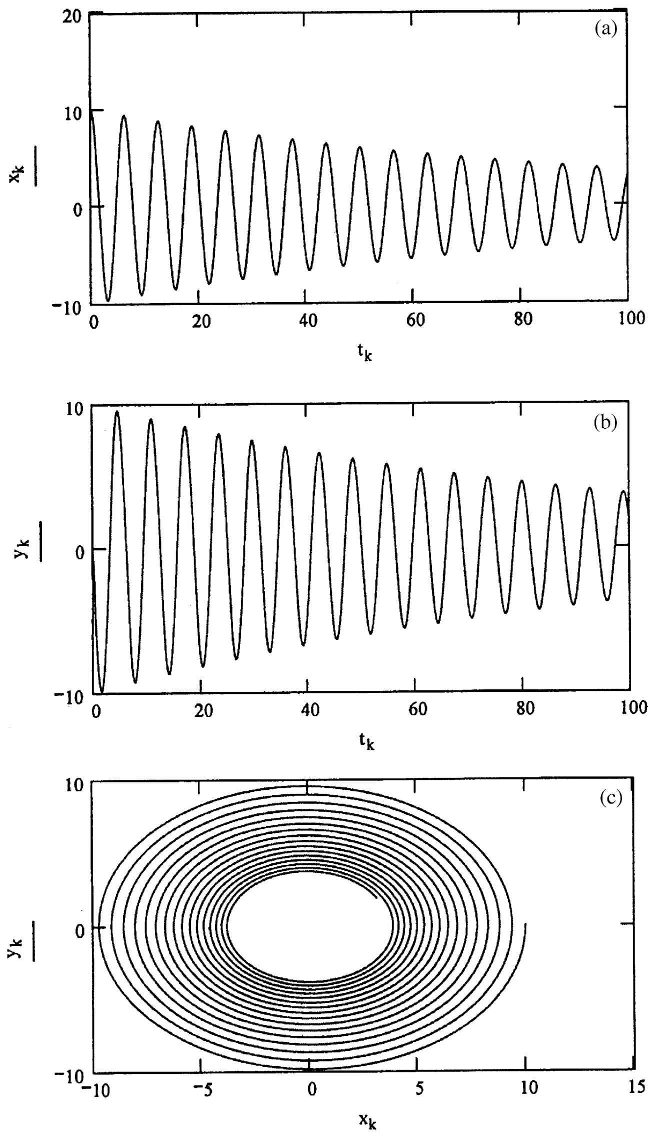

Fig. 1. (a) and (b) Plots of x_k and y_k versus t_k, and (c) x_k versus y_k, for $x_0 = 10.00$, $y_0 = 0.01$, $h = 0.001$, and $\varepsilon = 0.01$.

course, all of these results are in agreement with the above perturbation calculations based on MSVAP.

In summary, the analytical and numerical studies clearly indicate that the I–M nonlinear differential equation has all solutions of the form given by damped oscillations. This implies that the fixed-point, $(\bar{x}, \bar{y}) = (0, 0)$, is stable.

From a geometrical viewpoint, the trajectories in phase-space have the same properties as the linear damped harmonic oscillator [2]. A physical way of understanding this issue is to examine the equation

$$\frac{d^2x}{dt^2} + 2\varepsilon\frac{dx}{dt} + x = -2^{4/3}\varepsilon\left(\frac{dx}{dt}\right)^{2/3} \tag{28}$$

under the transformation $t \to -t$; doing this gives

$$\frac{d^2x}{dt^2} - 2\varepsilon\frac{dx}{dt} + x = -2^{4/3}\varepsilon\left(\frac{dx}{dt}\right)^{2/3}. \tag{29}$$

Note that every term retains its original sign except for the linear damping term. Consequently, in spite of a more complex dynamics expected for Eq. (29), its solutions should be damped because of the dissipative effects arising from the linear damping. This analysis also implies that the non-linear term on the right side of Eq. (29) does not give rise to any dissipative effects with regard to the full differential equation. Examples of this behavior are well known; see Section 2.4.2 of M [2].

Future work on this equation will concentrate on two issues: (1) Use phase-space and related methods to mathematically show that the fixed-point, $(\bar{x}, \bar{y}) = (0, 0)$, is globally stable; and (2) try to construct a perturbation procedure to calculate higher order terms that provide a uniformly valid approximation to the solution.

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